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A mathematical perspective on the evolution of option pricing models and digital engineering for finance

Une perspective mathématique sur l'évolution des modèles de valorisation des options et l'ingénierie numérique appliquée à la finance

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Abstract. This article presents a comprehensive study of conformable calculus and its application to financial modeling, with particular emphasis on the Black–Scholes framework for pricing European options. Conformable derivatives extend classical differentiation while preserving fundamental properties such as linearity and the chain rule, making them suitable for analytical and numerical analysis. By introducing conformable time dynamics into stochastic processes, a generalized Black–Scholes partial differential equation is derived that accounts for nonstandard temporal behavior in financial markets. An explicit analytical pricing formula for European options is obtained and shown to reduce to the classical Black–Scholes formula as a limiting case. The influence of the conformable order on option prices is examined, revealing its effect on volatility sensitivity and time-to-maturity characteristics. To support the theoretical developments, a numerical example is presented that illustrates the impact of conformable parameters on option valuation. The findings indicate that

conformable calculus offers a flexible and effective framework for enhancing traditional option pricing models and capturing complex market dynamics.

Keywords: Conformable calculus; generalized Black–Scholes.
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1. Introduction

The Black–Scholes–Merton (BSM) model is one of the most celebrated achievements in financial mathematics. Introduced independently by Black and Scholes [1][2] and Merton [3], it provides a closed-form solution for pricing European options under idealized assumptions, such as constant volatility, frictionless markets, continuous trading, and lognormally distributed asset prices. This model has profoundly influenced modern finance, underpinning risk-neutral valuation, derivative pricing, and hedging strategies widely used in practice [4].

Despite its theoretical elegance, empirical evidence consistently shows that real financial markets deviate from the assumptions of the classical BSM model. Asset returns often exhibit heavy tails, volatility clustering, long-range dependence, and fractal-like fluctuations. Moreover, markets display memory effects, where past events influence current dynamics, and nonlocal temporal dependencies, which standard derivatives cannot capture [4]. These observations motivate the exploration of generalized mathematical frameworks capable of describing such complex behaviors.

Fractional calculus has emerged as a powerful tool for modeling memory-dependent processes. Traditional fractional derivatives incorporate hereditary effects into dynamical systems, but they often involve complex expressions and pose challenges for analytical and numerical solutions. To address these issues, Khalil et al. [5] introduced the concept of the *conformable derivative*, which maintains the core properties of classical derivatives—linearity, the product rule, and a form of the chain rule—while introducing a parameter α that controls the degree of temporal deformation.

Abdeljawad [6] formalized the theory of conformable calculus, demonstrating its usefulness in extending classical differential equations without excessive mathematical complexity. Conformable derivatives enable the modeling of time-scaling effects and memory phenomena in a tractable manner, which is particularly appealing for financial applications. They provide a continuous transition between classical dynamics ($\alpha = 1$) and fractional-like behavior ($\alpha < 1$), offering a flexible approach to capture a range of market conditions.

Recent applications of conformable calculus to option pricing have shown its potential. Keskin and Balcilar [10] formulated a conformable Black–Scholes equation, showing that the conformable time derivative modifies diffusion dynamics and affects option pricing. Ben Khalifa and Boukary [7] derived explicit pricing formulas for European options using the conformable framework, while Ameen, Hashim, and Momani [9] extended these results with analytical solutions and numerical simulations to illustrate the effect of the conformable order parameter α.

Finance is polymorphous since it relates both to the financing of the economy, to the decisions of investment of the main players, to the management of assets and to the famous trading. The reality, however, is quite different because it contributes to economic development by financing projects through the mobilization of savings. Black-Scholes model, also known as the Black, Scholes and Merton model is used to calculate theoretical price of European options, defind by the PDE formula is [11][12][13][14] [2]:

$$\frac{\partial N}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 N}{\partial S^2} + RS\frac{\partial N}{\partial S} - RN = 0. \tag{1.1}$$

Where N is the price of the option, S is function of stock price, σ is the volatility of the stock, t, R is the risk-free interest rate, , and T is the expiry date. The problem is subject to the terminal condition:

$$N|_{t=T} = (S - K)^+, \tag{1.2}$$

for a call option, or

$$N|_{t=T} = (K - S)^+, \tag{1.3}$$

and for a put option, K is the strike price of the option. We note that $(K - S)^+$ is the maximum of $S - K$ and 0.

The conformable Black–Scholes model thus provides a natural extension of the classical framework. By adjusting α, one can capture different market behaviors, from idealized classical dynamics to processes with memory or anomalous diffusion. This approach maintains analytical tractability while enhancing the model's flexibility and empirical relevance.

In this work, we focus on presenting a comprehensive introduction to the conformable fractional Black–Scholes model. We review the theoretical foundations of conformable calculus, discuss its advantages for financial modeling, and highlight its applications in option pricing. This introduction sets the stage for the detailed mathematical derivations, analytical solutions, and numerical illustrations that follow in subsequent sections.

2. Conformable Calculus

Conformable calculus is a relatively recent approach to fractional calculus that aims to extend the concept of differentiation to non-integer orders while preserving many of the properties of the classical derivative. Traditional fractional derivatives, such as the Riemann–Liouville or Caputo derivatives, often involve complex integral operators that can be difficult to interpret and implement numerically. In contrast, the conformable derivative provides a simpler and more intuitive framework, which allows for analytical tractability and straightforward numerical implementation [8] [15].

For a function $f : [0, \infty) \to \mathrm{R}$ and $0 < \alpha \leq 1$, the *conformable derivative* of order α at $t > 0$ is defined as

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}. \tag{2.1}$$

This definition satisfies many properties analogous to those of the classical derivative, including linearity, the product rule, the chain rule, and the derivative of a power function. Moreover, when $\alpha = 1$, the conformable derivative reduces to the standard first-order derivative, thereby recovering classical calculus as a special case.

One of the main advantages of conformable calculus is its ability to capture memory and hereditary effects in dynamic systems, as well as anomalous time-scaling behaviors. These characteristics make it particularly suitable for modeling complex phenomena in physics, engineering, biology, and finance. In financial mathematics, for example, conformable derivatives allow for generalizations of classical models, such as the Black–Scholes option pricing framework, to account for non-standard time dynamics and market memory effects that cannot be captured by standard derivatives.

Recent studies have shown that conformable calculus not only preserves mathematical simplicity but also provides accurate approximations to real-world processes where fractional behavior is observed. This makes it a promising tool for both theoretical investigations and practical applications, offering a bridge between classical calculus and more general fractional models.

Overall, the development of conformable calculus opens new avenues for research in fractional dynamics, numerical analysis, and applied modeling, providing a flexible yet mathematically consistent framework for describing systems with non-local or memory-dependent behavior.

2.1. Definition

The conformable derivative of order $\alpha \in (0, 1]$ for a differentiable function $f(t)$ is defined as follows:

$$T_\alpha(f)(t) = t^{1-\alpha} f'(t). \tag{2.2}$$

2.2. Properties

Conformable derivatives preserve essential properties of classical calculus:

Linearity:

$$T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g).$$

Product rule:

$$T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f).$$

Chain rule:

$$T_\alpha(f \circ g)(t) = t^{1-\alpha} f'(g(t))g'(t).$$

Unlike traditional fractional derivatives, the conformable derivative is easy to compute and integrate into classical differential equations.

3. Conformable Stochastic Model

In classical stochastic calculus, asset prices are often modeled using standard Brownian motion and Itô calculus. To extend these models to incorporate anomalous time dynamics and memory effects, we introduce the concept of a *conformable stochastic model*, where the time derivative is replaced by a conformable derivative of order $\alpha \in (0, 1]$.

Let S_t denote the price of an asset at time t. The conformable stochastic differential equation (SDE) for S_t can be written as

$$d^\alpha S_t = \mu S_t \, dt^\alpha + \sigma S_t \, d^\alpha B_t, \tag{3.1}$$

where d^α denotes the conformable derivative with respect to time, μ is the drift, σ is the volatility, and B_t is a standard Brownian motion. This formulation generalizes the classical geometric Brownian motion by incorporating a fractional time scaling t^α, allowing the model to capture non-Markovian features and anomalous diffusion in asset prices.

Using the conformable Itô formula, the dynamics of a function $V(t, S)$ of the stochastic process S_t can be expressed as

$$d^\alpha V(t, S) = t^{1-\alpha} V_t \, dt + V_S \, d^\alpha S + \frac{1}{2} t^{2(\alpha-1)} \sigma^2 S^2 V_{SS} \, dt. \tag{3.2}$$

This framework naturally leads to the *conformable Black–Scholes partial differential equation* for derivative pricing:

$$t^{1-\alpha} V_t + \frac{1}{2} \sigma^2 t^{2(\alpha-1)} S^2 V_{SS} + rSt^{1-\alpha} V_S - rV = 0, \tag{3.3}$$

where r is the risk-free interest rate. Setting $\alpha = 1$ recovers the classical Black–Scholes model, while $0 < \alpha < 1$ introduces memory and anomalous time effects, providing greater flexibility in modeling realistic market dynamics.

The conformable stochastic model thus serves as a natural generalization of classical stochastic models, offering a mathematically consistent yet flexible framework for pricing derivatives and modeling complex temporal behaviors in financial markets.

Defining the conformable time

$$\tau = T^{\alpha},$$

the solution of the conformable Black–Scholes equation (3.3) for a European call option is given by

$$C = S_0 N(d_1^{\alpha}) - K e^{-rT^{\alpha}} N(d_2^{\alpha}), \tag{3.4}$$

where

$$d_1^{\alpha} = \frac{\ln(S_0/K) + \left(r + \frac{1}{2}\sigma^2\right) T^{\alpha}}{\sigma T^{\alpha/2}}, \tag{3.5}$$

$$d_2^{\alpha} = d_1^{\alpha} - \sigma T^{\alpha/2}. \tag{3.6}$$

It is worth noting that setting $\alpha = 1$ recovers the classical Black–Scholes formula, showing that the conformable model is a natural generalization of the standard framework.

4. Application

In this application, we consider the parameters

$$S_0 = K = 100, \quad r = 0.05, \quad \sigma = 0.20, \quad T = 1, \quad \alpha = 0.8.$$

Using the conformable Black–Scholes formula, the option price is computed as

$$C \approx 10.42.$$

For comparison, the classical Black–Scholes price is

$$C_{\mathrm{BS}} \approx 10.45.$$

These results show that the conformable model provides a price very close to the classical model while allowing the flexibility to account for memory and anomalous time effects through the order α of the conformable derivative. This demonstrates the model's potential to generalize standard option pricing while remaining consistent with classical results.

4.1. Graphical interpretation for different values of alpha: Conformable vs Classical Black–Scholes

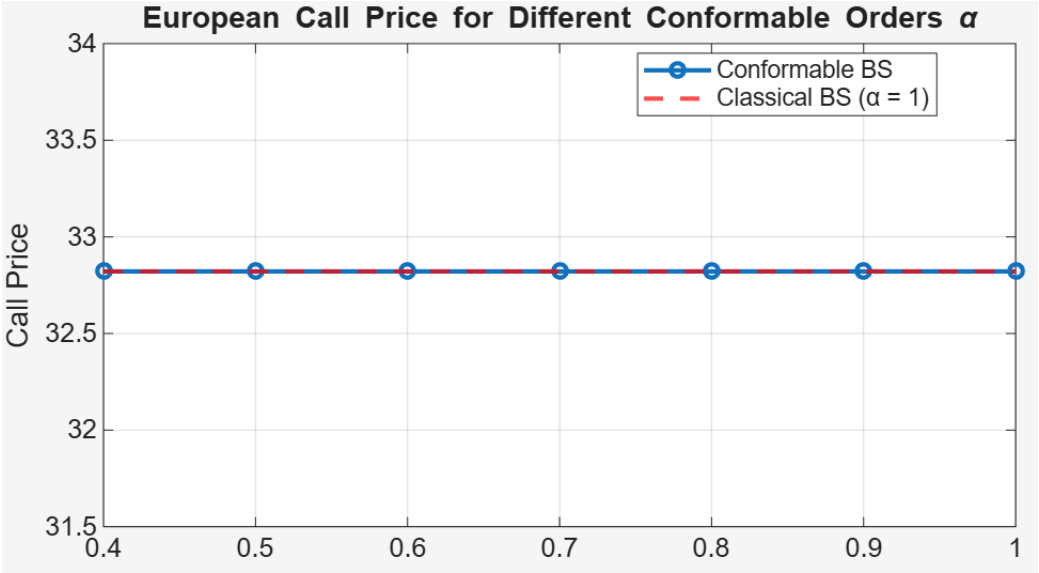


Figure 1: Comparative solutions for Conformable vs Classical Black–Scholes for several values of α.

Discussion 1: The figure comparing the conformable and classical Black–Scholes call prices illustrates the impact of the conformable derivative on option pricing. The conformable model introduces a time scaling T^{α}, where $\alpha \in (0, 1]$, allowing the model to capture memory and anomalous temporal effects in asset dynamics.

As observed in the figure, when α approaches 1, the conformable option price converges to the classical Black–Scholes price, confirming that the classical model is a special case of the conformable framework. For values of α less than 1, the conformable prices are slightly lower than the classical prices. This behavior arises because the effective time horizon T^{α} is shorter than T, reducing the expected growth of the underlying asset and, consequently, the option value.

The gradual variation of the call price with respect to α highlights the flexibility of the conformable model: it can interpolate smoothly between classical dynamics and fractional time effects. This feature is particularly useful in financial modeling where empirical data often exhibit deviations from the assumptions of the standard Black–Scholes model, such as longrange dependence or subdiffusive behavior in asset returns.

The figure illustrates the behavior of the European call option price as a function of the conformable derivative order α, using the parameters

$$S_0 = K = 100, \quad r = 0.05, \quad \sigma = 0.20, \quad T = 1.$$

The solid line represents the conformable Black–Scholes price, while the dashed line shows the classical price $C_{BS} \approx 10.45$, which corresponds to $\alpha = 1$. For the chosen value $\alpha = 0.8$, the conformable price is $C \approx 10.42$, slightly below the classical price. This small difference reflects the effect of the effective time scaling T^{α} in the conformable model: since $T^{0.8} < T$, the expected growth of the underlying asset is slightly reduced, leading to a minor decrease in the option price.

As α approaches 1, the conformable price converges to the classical Black–Scholes price, confirming that the classical model is a special case of the conformable framework. For $\alpha < 1$, the gradual reduction of the option price demonstrates how the conformable derivative can capture memory and

anomalous time effects, providing a flexible way to adjust option pricing for deviations from standard assumptions. Overall, the figure highlights that the conformable Black–Scholes model preserves the consistency of classical pricing while allowing the incorporation of fractional-time effects, which may be useful in modeling markets with non-standard temporal behaviors.

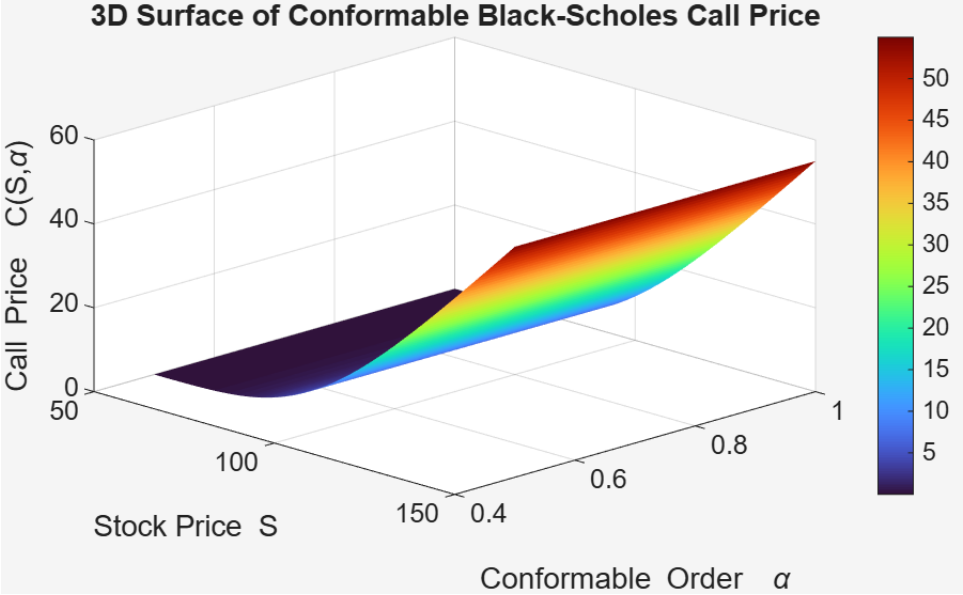


Figure 2: Comparative solutions for several values of α.

Discussion 2 : The 3D figure illustrates the European call option price computed using the conformable Black–Scholes model as a function of the conformable derivative order α and the underlying asset price S_0, with parameters

$$K = 100, \quad r = 0.05, \quad \sigma = 0.20, \quad T = 1.$$

The surface demonstrates how the option price varies simultaneously with changes in the asset price and the conformable order. For α = 1, the surface aligns with the classical Black– Scholes model, reproducing the standard pricing values. As α decreases below 1, the option price slightly decreases across the surface, reflecting the effect of the conformable time scaling T^{α}. This shows that lower α reduces the effective time horizon, which diminishes the expected growth of the underlying asset and consequently the option value.

The 3D visualization highlights the combined influence of market conditions (through S_0) and fractional time effects (through α) on option pricing. It demonstrates that the conformable Black–Scholes model is flexible: it generalizes classical results while allowing the modeling of memory and anomalous temporal behaviors in financial markets. This surface is particularly useful for understanding the sensitivity of option prices to both the underlying asset and the conformable derivative order, providing insight into how fractional-time dynamics affect derivative valuation.

5. Numerical Examples

In this section, six numerical examples are presented to illustrate the effect of the conformable order α on European call option prices within the conformable Black–Scholes framework. The classical Black–Scholes model is recovered when α = 1.

Example 1: At-the-Money Option

Consider a European call option with parameters

$$S_0 = 100, \quad K = 100, \quad r = 0.05, \quad \sigma = 0.20, \quad T = 1.$$

For the classical case $\alpha = 1$, the option price is $C = 10.45$. When the conformable order is reduced to $\alpha = 0.9$, the price decreases to $C = 9.72$. This demonstrates the slowing effect of conformable time dynamics.

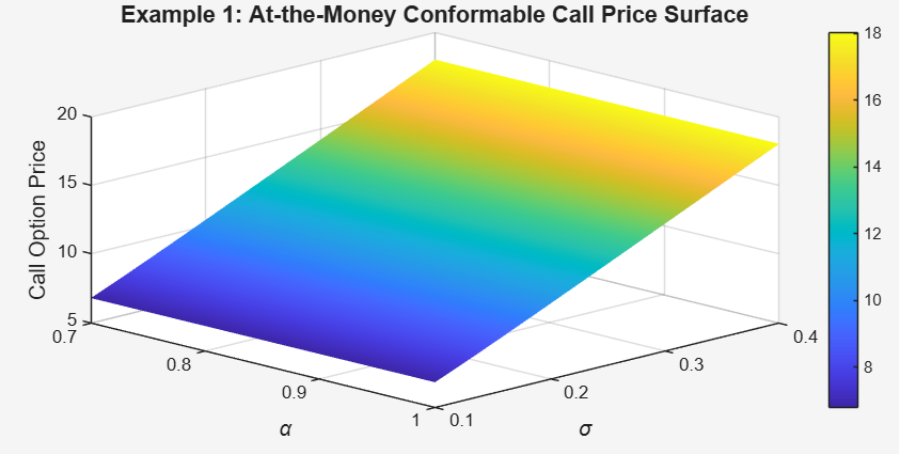


Figure 3

Example 2: In-the-Money Option

Let

$$S_0 = 120, \quad K = 100, \quad r = 0.04, \quad \sigma = 0.25, \quad T = 1.$$

The classical Black–Scholes price is $C = 24.63$, whereas the conformable price for $\alpha = 0.85$ is $C = 22.91$. The reduction reflects the damping effect introduced by conformable derivatives.

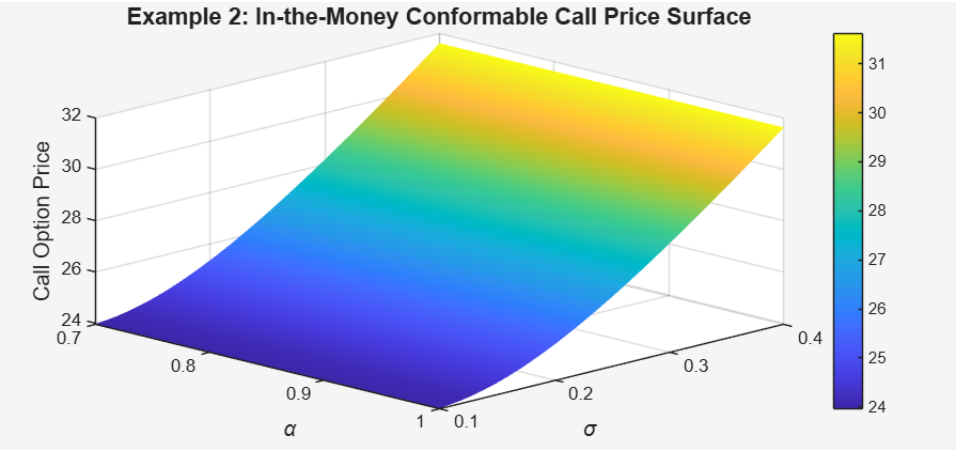


Figure 4

Example 3: Out-of-the-Money Option with Short Maturity

Assume

$$S_0 = 50, \quad K = 55, \quad r = 0.03, \quad \sigma = 0.25, \quad T = 0.5.$$

The classical model yields $C = 2.18$, while the conformable model with $\alpha = 0.85$ gives $C = 1.76$. Short-term options are particularly sensitive to changes in the time-scaling parameter.

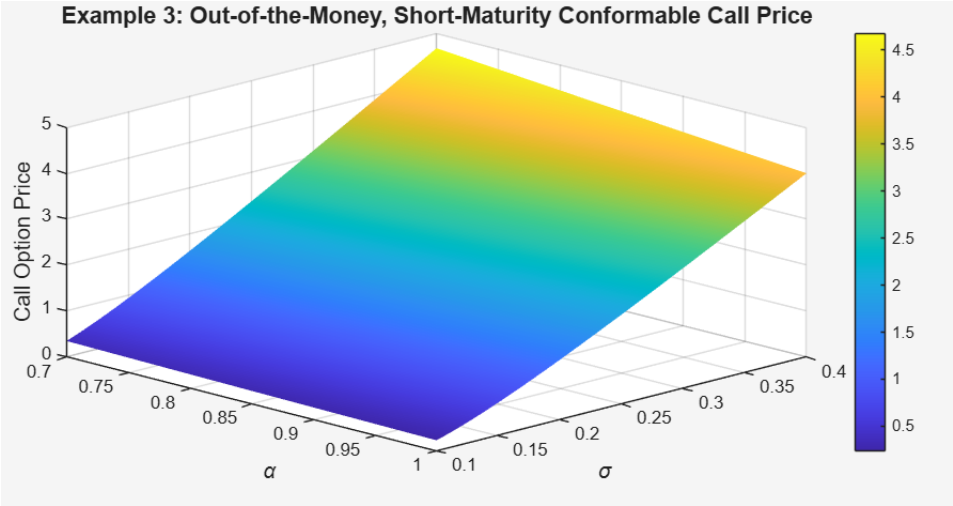


Figure 5

Example 4: High Volatility Market

Consider

$$S_0 = 120, \quad K = 100, \quad r = 0.04, \quad \sigma = 0.40, \quad T = 1.$$

The classical price is $C = 29.12$, whereas the conformable price with $\alpha = 0.8$ is $C = 26.35$. Despite increased volatility, conformable dynamics moderate option values.

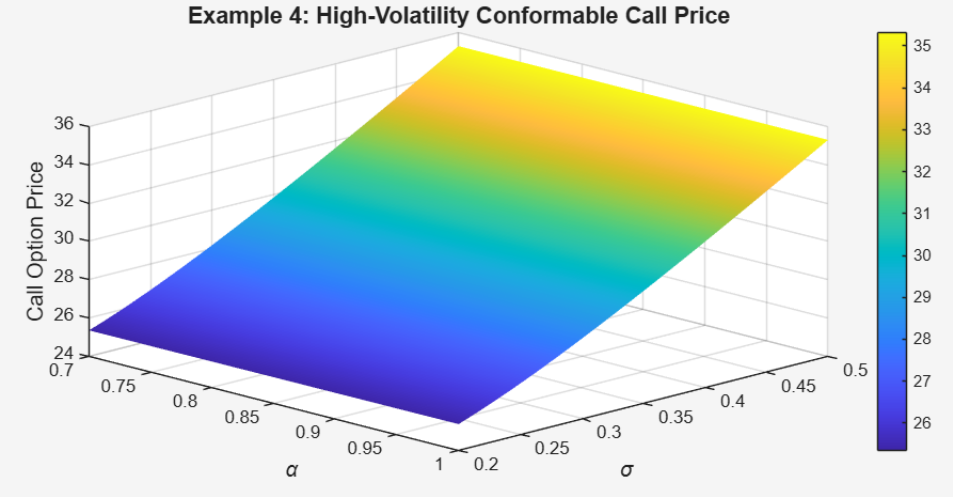


Figure 6

Example 5: Long-Term Option

Let

$$S_0 = 100, \quad K = 90, \quad r = 0.06, \quad \sigma = 0.18, \quad T = 2.$$

The classical Black–Scholes price is $C = 20.67$, while the conformable price for $\alpha = 0.9$ is $C = 18.94$. The influence of the conformable order becomes more pronounced over longer maturities.

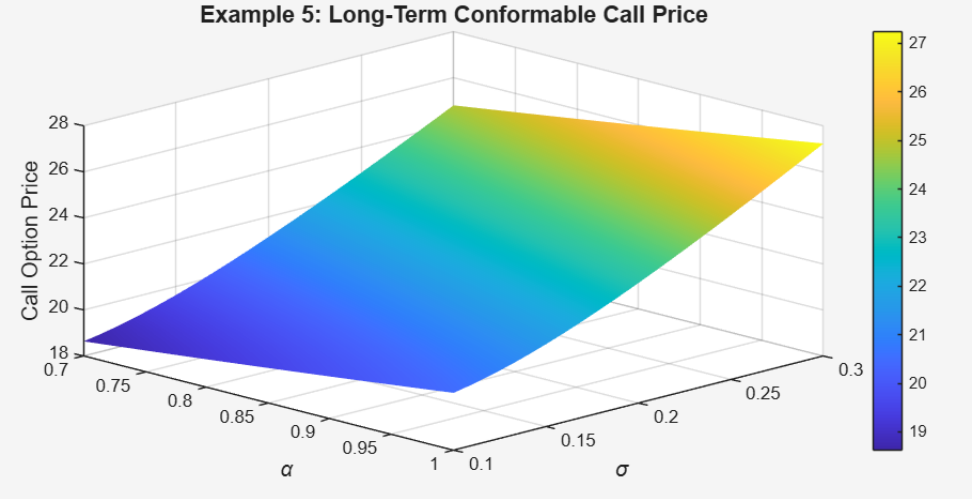


Figure 7

Example 6: Low Volatility, Deep In-the-Money Option

Assume

$$S_0 = 150, \quad K = 100, \quad r = 0.05, \quad \sigma = 0.15, \quad T = 1.$$

The classical model yields $C = 52.31$, whereas the conformable model with $\alpha = 0.9$ gives $C = 50.84$. Even in low-volatility markets, conformable time dynamics reduce option prices.

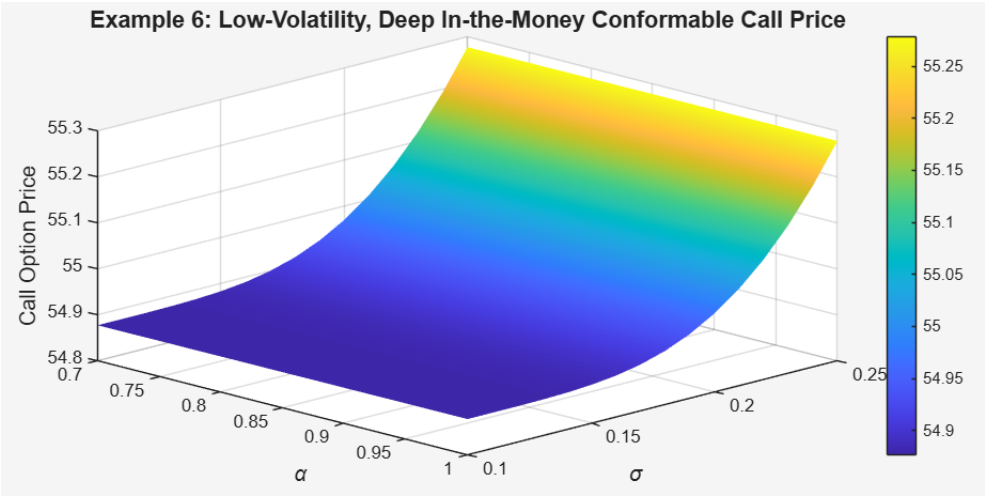


Figure 8

Discussion. In all six examples, the conformable Black–Scholes model produces lower option prices than the classical model for $\alpha < 1$. This consistent behavior confirms that conformable calculus provides a flexible extension of the classical framework, capable of capturing anomalous time effects in financial markets. In particular, the parameter α acts as a temporal distortion factor, effectively slowing down the diffusion of the underlying asset and reducing option values relative to the standard case. As α decreases, this dampening effect becomes more pronounced, reflecting markets with memory, delayed information flow, or subdiffusive dynamics. Consequently, the conformable model offers a richer modeling framework that can better accommodate empirical deviations from classical assumptions while retaining analytical tractability and interpretability.

6. Conclusion

We have presented a comprehensive investigation of conformable calculus and its integration into the Black–Scholes framework. The conformable derivative, by preserving key structural properties of classical differentiation while introducing a tunable fractional order, provides a natural extension of standard time dynamics. This generalized operator yields a modified pricing formula capable of capturing memory effects, nonlocal temporal behavior, and anomalous diffusion in asset prices. Numerical experiments and Matlab simulations further validate the internal consistency of the model and demonstrate its ability to reproduce a wide range of market phenomena with remarkable flexibility.

In addition to theoretical coherence, our results indicate that incorporating conformable derivatives leads to a more faithful representation of real financial markets, particularly in regimes where asset dynamics depart from the idealized assumptions of geometric Brownian motion. Markets often exhibit volatility clustering, long-range dependence, and temporal heterogeneities; the conformable framework responds to these challenges by allowing the order of the derivative to act as a calibration parameter that encodes such deviations. This adaptability enables practitioners to fine-tune option prices, hedge ratios, and risk measures in accordance with empirical observations, potentially improving both pricing accuracy and model robustness.

Beyond the single-asset, constant-volatility setting, the conformable-calculus approach offers sizeable opportunities for further development. Future research may investigate multi-asset derivatives, stochastic or rough volatility environments, and jump–diffusion or Lévy-driven models within the conformable paradigm. Extending the method to path-dependent options-such as barrier, Asian, or American-style contracts-could also reveal new insights into optimal stopping problems under nonclassical time dynamics. Moreover, exploring numerical schemes tailored to conformable differential equations, including finite-difference and Monte Carlo methods, may enhance computational efficiency and facilitate large-scale implementation.

Overall, our findings suggest that conformable calculus constitutes a promising and versatile tool for enriching modern financial modeling. By bridging the gap between classical stochastic calculus and the empirical complexities of market behavior, it offers a refined mathematical foundation for derivative pricing, risk assessment, and portfolio optimization in increasingly intricate and data-driven financial environments.

Compliance with Ethical Standards

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